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A PROOF OF THE STRONG NO LOOP CONJECTURE

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Abstract. The strong no loop conjecture states that a simple module of finite projective dimension over an artin algebra has no non-zero self-extension. The main result of this paper establishes this well known conjecture for finite dimensional algebras over an algebraically closed field.

Introduction

Let Λ be an artin algebra, and denote by modΛ the category of finitely generated right Λ-modules. It is an important problem in the representation theory of algebras to determine whether Λ has finite or infinite global dimension, and more specifically, whether a simple Λ-module has finite or infinite projective dimension. For instance, the derived category $D^b(\text{mod}\Lambda)$ has Auslander-Reiten triangles if and only if Λ has finite global dimension; see [7, 8]. One approach to this problem is to consider the extension quiver of Λ, which has vertices given by a complete set of non-isomorphic simple Λ-modules and single arrows $S \to T$, where S and T are vertices such that $\text{Ext}^1_\Lambda(S,T)$ is non-zero. Then the *no loop conjecture* affirms that the extension quiver of Λ contains no loop if Λ is of finite global dimension, while the *strong no loop conjecture*, which is due to Zacharia, strengthens this to state that a vertex in the extension quiver admits no loop if it has finite projective dimension; see [1, 10].

The no loop conjecture was first explicitly established for artin algebras of global dimension two; see [5]. For finite dimensional elementary algebras, as shown in [10], this can be easily derived from an earlier result of Lenzing on Hochschild homology in [13]. Lenzing's technique was to extend the notion of the trace of endomorphisms of projective modules, defined by Hattori and Stallings in [9, 18], to endomorphisms of modules over a noetherian ring with finite global dimension, and apply it to a particular kind of filtration for the regular module.

In contrast, up to now, the strong no loop conjecture has only been verified for some special classes of algebras such as monomial algebras; see [2, 10], special biserial algebras; see [14], and algebras with at most two simple modules and radical cubed zero; see [12]. Many other partial results can be found in [3, 4, 6, 15, 16, 20]. Most recently, Skorodumov generalized and localized Lenzing's filtration to indecomposable projective modules. This allowed him to prove this conjecture for finite dimensional elementary algebras of finite representation type; see [17].

In this paper, we shall localize Lenzing's trace function to endomorphisms of modules in modΛ with an e-bounded projective resolution, where e is an idempotent in Λ. The key point is that every module in modΛ has an e-bounded projective resolution if the semi-simple module supported by e has finite injective dimension. This will enable us to solve the strong no loop conjecture for a large class of artin algebras including finite dimensional elementary algebras, and particularly, for finite dimensional algebras over an algebraically closed field.

1. Localized trace function and Hochschild homology

Throughout, J will stand for the Jacobson radical of Λ. The additive subgroup of Λ generated by the elements $ab - ba$ with $a, b \in \Lambda$ is called the *commutator group* of Λ and written as $[\Lambda, \Lambda]$. One defines then the Hochschild homology group $\mathrm{HH}_0(\Lambda)$ to be $\Lambda/[\Lambda, \Lambda]$. We shall say that $\mathrm{HH}_0(\Lambda)$ is *radical-trivial* if $J \subseteq [\Lambda, \Lambda]$.

To start with, we recall the notion of the trace of an endomorphism φ of a projective module P in $\mathrm{mod}\Lambda$, as defined by Hattori and Stallings in [9, 18]; see also [10, 13]. Write $P = e_1\Lambda \oplus \cdots \oplus e_r\Lambda$, where the e_i are primitive idempotents in Λ. Then $\varphi = (a_{ij})_{r\times r}$, where $a_{ij} \in e_i\Lambda e_j$. The *trace* of φ is defined to be

$$\mathrm{tr}(\varphi) = \sum\nolimits_{i=1}^{r} a_{ii} + [\Lambda, \Lambda] \in \mathrm{HH}_0(\Lambda).$$

We collect some well known properties of this trace function in the following proposition, in which the property (2) is the reason for defining the trace to be an element in $\mathrm{HH}_0(\Lambda)$.

1.1. Proposition (Hattori-Stallings). *Let P, P' be projective modules in* $\mathrm{mod}\Lambda$.
(1) *If $\varphi, \psi \in \mathrm{End}_\Lambda(P)$, then* $\mathrm{tr}(\varphi + \psi) = \mathrm{tr}(\varphi) + \mathrm{tr}(\psi)$.
(2) *If $\varphi : P \to P'$ and $\psi : P' \to P$ are Λ-linear, then* $\mathrm{tr}(\varphi\psi) = \mathrm{tr}(\psi\varphi)$.
(3) *If $\varphi = (\varphi_{ij})_{2\times 2} : P \oplus P' \to P \oplus P'$, then* $\mathrm{tr}(\varphi) = \mathrm{tr}(\varphi_{11}) + \mathrm{tr}(\varphi_{22})$.
(4) *If $\psi : P \to P'$ is an isomorphism and $\varphi \in \mathrm{End}_\Lambda(P)$, then* $\mathrm{tr}(\psi\varphi\psi^{-1}) = \mathrm{tr}(\varphi)$.
(5) *If $\varphi : \Lambda \to \Lambda$ is the left multiplication by $a \in \Lambda$, then* $\mathrm{tr}(\varphi) = a + [\Lambda, \Lambda]$.

Next, we recall Lenzing's extension of this notion to endomorphisms of modules of finite projective dimension. For $M \in \mathrm{mod}\Lambda$, let $\mathcal{P}_M$ denote a projective resolution

$$\cdots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

of M in $\mathrm{mod}\Lambda$. For each $\varphi \in \mathrm{End}_\Lambda(M)$, one can construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_i \arrow[r, "d_i"] \arrow[d, "\varphi_i"] & P_{i-1} \arrow[r] \arrow[d, "\varphi_{i-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\varphi_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\varphi_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & P_i \arrow[r, "d_i"] & P_{i-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

in $\mathrm{mod}\Lambda$. We shall call $\{\varphi_i\}_{i\geq 0}$ a *lifting* of φ to $\mathcal{P}_M$. If M is of finite projective dimension, then one may assume that $\mathcal{P}_M$ is bounded and define the *trace* of φ by

$$\mathrm{tr}(\varphi) = \sum\nolimits_{i=0}^{\infty} (-1)^i \,\mathrm{tr}(\varphi_i) \in \mathrm{HH}_0(\Lambda),$$

which is independent of the choice of $\mathcal{P}_M$ and $\{\varphi_i\}$; see [13], and also [10].

Our strategy is to localize this construction. Let e be an idempotent in Λ. Set

$$\Lambda_e = \Lambda/\Lambda(1 - e)\Lambda.$$

The canonical algebra projection $\Lambda \to \Lambda_e$ induces a group homomorphism

$$H_e : \mathrm{HH}_0(\Lambda) \to \mathrm{HH}_0(\Lambda_e).$$

For an endomorphism φ of a projective module in $\mathrm{mod}\Lambda$, we define its *e-trace* by

$$\mathrm{tr}_e(\varphi) = H_e(\mathrm{tr}(\varphi)) \in \mathrm{HH}_0(\Lambda_e).$$

It is evident that this e-trace function has the properties (1) to (4) stated in Proposition 1.1. More importantly, we have the following result.

1.2. Lemma. *Let e be an idempotent in Λ, and let P be a projective module in* $\mathrm{mod}\Lambda$ *whose top is annihilated by e. If $\varphi \in \mathrm{End}_\Lambda(P)$, then $\mathrm{tr}_e(\varphi) = 0$.*

Proof. We may assume that P is non-zero. Then $1 - e = e_1 + \cdots + e_r$, where the e_i are pairwise orthogonal primitive idempotents in Λ. Let $\varphi \in \mathrm{End}_\Lambda(P)$. By Proposition 1.1(3), we may assume that P is indecomposable. Then $P \cong e_s\Lambda$ for some $1 \le s \le r$. By Proposition 1.1(4), we may assume that $P = e_s\Lambda$. Then φ is the left multiplication by some $a \in e_s\Lambda e_s$. By Proposition 1.1(5),

$$\mathrm{tr}_e(\varphi) = H_e(a + [\Lambda, \Lambda]) = \bar{a} + [\Lambda_e, \Lambda_e],$$

where $\bar{a} = a + \Lambda(1 - e)\Lambda$. Since $a = e_s a e_s = (1 - e)a(1 - e) \in \Lambda(1 - e)\Lambda$, we get $\mathrm{tr}_e(\varphi) = 0$. The proof of the lemma is completed.

To extend the e-trace function, we shall call a projective resolution $\mathcal{P}_M$ of M *e-bounded* if e annihilates the tops of all but finitely many terms in $\mathcal{P}_M$. In this case, if $\varphi \in \mathrm{End}_\Lambda(M)$ with a lifting $\{\varphi_i\}_{i\ge 0}$ to $\mathcal{P}_M$ then, by Lemma 1.2, $\mathrm{tr}_e(\varphi_i) = 0$ for all but finitely many i. This allows us to define the *e-trace* of φ by

$$\mathrm{tr}_e(\varphi) = \sum\nolimits_{i=0}^{\infty} (-1)^i\, \mathrm{tr}_e(\varphi_i) \in \mathrm{HH}_0(\Lambda_e).$$

1.3. Lemma. *Let e be an idempotent in Λ. The e-trace is well defined for endomorphisms of modules in* $\mathrm{mod}\Lambda$ *having an e-bounded projective resolution.*

Proof. Let M be a module in $\mathrm{mod}\Lambda$ having an e-bounded projective resolution

$$\mathcal{P}_M : \qquad \cdots \to P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0.$$

Fix $\varphi \in \mathrm{End}_\Lambda(M)$. We first show that $\mathrm{tr}_e(\varphi)$ is independent of the choice of its lifting to $\mathcal{P}_M$. By Proposition 1.1(1), it amounts to proving that $\sum_{i=0}^{\infty}(-1)^i\, \mathrm{tr}_e(\varphi_i) = 0$ for any lifting $\{\varphi_i\}_{i\ge 0}$ of the zero endomorphism of M. Indeed, let $h_i : P_i \to P_{i+1}$ be morphisms such that $\varphi_0 = d_1 h_0$ and $\varphi_i = d_{i+1}h_i + h_{i-1}d_i$, for $i \ge 1$. Applying Proposition 1.1, we get

$$\mathrm{tr}_e(\varphi_i) = \mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(h_{i-1}d_i) = \mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(d_i h_{i-1}),$$

for $i \ge 1$. On the other hand, by assumption, there exists some $m \ge 0$ such that e annihilates the top of P_i for $i \ge m$. By Lemma 1.2, $\mathrm{tr}_e(d_{m+1}h_m) = 0$ and $\mathrm{tr}_e(\varphi_i) = 0$ for $i \ge m$. This yields

$$\begin{aligned}
\sum\nolimits_{i=0}^{\infty}(-1)^i\, \mathrm{tr}_e(\varphi_i) &= \mathrm{tr}_e(\varphi_0) + \sum\nolimits_{i=1}^{m}(-1)^i\, \mathrm{tr}_e(\varphi_i)\\
&= \mathrm{tr}_e(d_1h_0) + \sum\nolimits_{i=1}^{m}(-1)^i\, (\mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(d_ih_{i-1}))\\
&= (-1)^m\, \mathrm{tr}_e(d_{m+1}h_m)\\
&= 0.
\end{aligned}$$

Next, we show that $\mathrm{tr}_e(\varphi)$ is independent of the choice of the e-bounded projective resolution $\mathcal{P}_M$. Suppose that M has another e-bounded projective resolution

$$\mathcal{P}'_M : \qquad \cdots \to P'_i \xrightarrow{d'_i} P'_{i-1} \to \cdots \to P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} M \to 0.$$

Considering φ, we get morphisms $u_i : P_i \to P'_i$ with $i \ge 0$ such that $d'_0u_0 = \varphi d_0$ and $d'_iu_i = u_{i-1}d_i$ for $i \ge 1$. Similarly, considering 1_M, we obtain maps $v_i : P'_i \to P_i$

with $i \geq 0$ such that $d_0 v_0 = d_0'$ and $d_i v_i = v_{i-1} d_i'$ for $i \geq 1$. Observe that $\{v_i u_i\}_{i\geq 0}$ and $\{u_i v_i\}_{i\geq 0}$ are liftings of φ to $\mathcal{P}_M$ and $\mathcal{P}_M'$, respectively. By Proposition 1.1(2), we have

$$\sum\nolimits_{i=0}^{\infty} (-1)^i \operatorname{tr}_e(u_i v_i) = \sum\nolimits_{i=0}^{\infty} (-1)^i \operatorname{tr}_e(v_i u_i).$$

The proof of the lemma is completed.

In the sequel, S_e will stand for the semi-simple Λ-module $e\Lambda/eJ$. Suppose that S_e has finite injective dimension. If M is a module in $\operatorname{mod}\Lambda$, then $\operatorname{Ext}^i_\Lambda(M, S_e) = 0$ for all sufficient large integers i, that is, the minimal projective resolution of M is e-bounded. Therefore, the e-trace is defined for every endomorphism in $\operatorname{mod}\Lambda$. In particular, if Λ is of finite global dimension, then we recover Lenzing's trace function by taking $e = 1_A$.

1.4. Proposition. *Let e be an idempotent in Λ. Consider a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "u"] \arrow[d, "\varphi_L"] & M \arrow[r, "v"] \arrow[d, "\varphi_M"] & N \arrow[r] \arrow[d, "\varphi_N"] & 0 \\ 0 \arrow[r] & L \arrow[r, "u"] & M \arrow[r, "v"] & N \arrow[r] & 0 \end{tikzcd}$$

in $\operatorname{mod}\Lambda$ *with exact rows. If L, N have e-bounded projective resolutions, then so does M and* $\operatorname{tr}_e(\varphi_M) = \operatorname{tr}_e(\varphi_L) + \operatorname{tr}_e(\varphi_N)$.

Proof. Assume that L and N have e-bounded projective resolutions as follows:

$$\mathcal{P}_L: \qquad \cdots \to P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} L \to 0$$

and

$$\mathcal{P}_N: \qquad \cdots \to P_i' \xrightarrow{d_i'} P_{i-1}' \to \cdots \to P_1' \xrightarrow{d_1'} P_0' \xrightarrow{d_0'} N \to 0.$$

By the Horseshoe lemma, there exists in $\operatorname{mod}\Lambda$ a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_i \arrow[r, "d_i"] \arrow[d, "q_i"] & P_{i-1} \arrow[r] \arrow[d, "q_{i-1}"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] \arrow[d, "q_0"] & L \arrow[r] \arrow[d, "u"] & 0 \\ \cdots \arrow[r] & P_i \oplus P_i' \arrow[r, "d_i''"] \arrow[d, "p_i"] & P_{i-1} \oplus P_{i-1}' \arrow[r] \arrow[d, "p_{i-1}"] & \cdots \arrow[r] & P_0 \oplus P_0' \arrow[r, "d_0''"] \arrow[d, "p_0"] & M \arrow[r] \arrow[d, "v"] & 0 \\ \cdots \arrow[r] & P_i' \arrow[r, "d_i'"] & P_{i-1}' \arrow[r] & \cdots \arrow[r] & P_0' \arrow[r, "d_0'"] & N \arrow[r] & 0 \end{tikzcd}$$

with exact rows, where $q_i = \binom{1}{0}$, $p_i = (0, 1)$ for all $i \geq 0$. In particular, the middle row is an e-bounded projective resolution of M which we denote by $\mathcal{P}_M$. Choose a lifting $\{f_i\}_{i\geq 0}$ of φ_L to $\mathcal{P}_L$ and a lifting $\{g_i\}_{i\geq 0}$ of φ_N to $\mathcal{P}_N$. It is well known; see, for example, [19, p. 46] that there exists a lifting $\{h_i\}_{i\geq 0}$ of φ_M to $\mathcal{P}_M$ such that

$$\begin{tikzcd} 0 \arrow[r] & P_i \arrow[r, "q_i"] \arrow[d, "f_i"] & P_i \oplus P_i' \arrow[r, "p_i"] \arrow[d, "h_i"] & P_i' \arrow[r] \arrow[d, "g_i"] & 0 \\ 0 \arrow[r] & P_i \arrow[r, "q_i"] & P_i \oplus P_i' \arrow[r, "p_i"] & P_i' \arrow[r] & 0 \end{tikzcd}$$

is commutative, for every $i \geq 0$. Since $h_i q_i = q_i f_i$ and $g_i p_i = p_i h_i$, we can write h_i as a (2×2)-matrix whose diagonal entries are f_i and g_i. Thus $\operatorname{tr}_e(h_i) = \operatorname{tr}_e(f_i) + \operatorname{tr}_e(g_i)$

by Proposition 1.1(3). As a consequence, $\operatorname{tr}_e(\varphi_M) = \operatorname{tr}_e(\varphi_N) + \operatorname{tr}_e(\varphi_L)$. The proof of the proposition is completed.

Finally, we shall describe the Hochschild homology group $\mathrm{HH}_0(\Lambda_e)$ in case S_e has finite injective dimension.

1.5. Theorem. *Let Λ be an artin algebra, and let e be an idempotent in Λ. If S_e has finite injective dimension, then $\mathrm{HH}_0(\Lambda_e)$ is radical-trivial.*

Proof. Suppose that S_e has finite injective dimension. Then the e-trace is defined for every endomorphism in $\operatorname{mod}\Lambda$. Let $x \in \Lambda$ be such that $\bar{x} = x + \Lambda(1-e)\Lambda$ lies in the radical of Λ_e, which is $(J + \Lambda(1-e)\Lambda)/\Lambda(1-e)\Lambda$. Hence, $\bar{x} = \bar{a}$ for some $a \in J$. Let $r > 0$ be such that $a^r = 0$, and consider the chain

$$0 = M_r \subseteq M_{r-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = \Lambda,$$

of submodules of Λ, where $M_i = a^i\Lambda$, $i = 0, \ldots, r$. Let $\varphi_0 : \Lambda \to \Lambda$ be the left multiplication by a. Since $\varphi_0(M_i) \subseteq M_{i+1}$, we see that φ_0 induces morphisms $\varphi_i : M_i \to M_i$, $i = 1, \ldots, r$, such that

$$\begin{tikzcd}
0 \arrow[r] & M_{i+1} \arrow[r] \arrow[d, "\varphi_{i+1}"] & M_i \arrow[r] \arrow[d, "\varphi_i"] & M_i/M_{i+1} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & M_{i+1} \arrow[r] & M_i \arrow[r] & M_i/M_{i+1} \arrow[r] & 0
\end{tikzcd}$$

commutes, and hence $\operatorname{tr}_e(\varphi_i) = \operatorname{tr}_e(\varphi_{i+1})$ by Proposition 1.4, for $i = 0, 1, \ldots, r-1$. Applying Proposition 1.1(5), we get

$$\bar{a} + [\Lambda_e, \Lambda_e] = H_e(a + [\Lambda, \Lambda]) = H_e(\operatorname{tr}(\varphi_0)) = \operatorname{tr}_e(\varphi_0) = \operatorname{tr}_e(\varphi_r) = 0,$$

that is, $\bar{x} = \bar{a} \in [\Lambda_e, \Lambda_e]$. The proof of the theorem is completed.

Taking $e = 1_A$, we recover the following well known result; see, for example, [13].

1.6. Corollary. *If Λ is an artin algebra of finite global dimension, then $\mathrm{HH}_0(\Lambda)$ is radical-trivial.*

Indeed, if Λ is a finite dimensional algebra of finite global dimension over a field of characteristic zero, then all the Hochschild homology groups $\mathrm{HH}_i(\Lambda)$ with $i \geq 1$ vanish; see [13]. However, in the situation as in Theorem 1.5, the higher Hochschild homology groups of Λ_e do not necessarily vanish and Λ_e may be of infinite global dimension.

Example. Let $\Lambda = kQ/I$, where k is a field, Q is the quiver

$$\begin{tikzcd}
1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & 2 \arrow[d, "\beta"] \\
4 \arrow[r, "\delta"] & 3 \arrow[ul, "\varepsilon"']
\end{tikzcd}$$

and I is the ideal in kQ generated by $\alpha\beta - \gamma\delta, \beta\varepsilon, \delta\varepsilon, \varepsilon\alpha$. One can show that Λ has finite global dimension. Now, let e be the sum of the primitive idempotents in Λ corresponding to the vertices $1, 2, 3$. Then Λ_e is a Nakayama algebra with radical squared zero, which clearly has infinite global dimension. By Theorem 1.5, $\mathrm{HH}_0(\Lambda_e)$ is radical-trivial. However, a direct computation shows that $\mathrm{HH}_2(\Lambda_e)$ is non-zero; see also [11].

2. Main results

The main objective of this section is to apply the previously obtained result to solve the strong no loop conjecture for finite dimensional algebras over an algebraically closed field. We start with an artin algebra Λ with a primitive idempotent e. We shall say that Λ is *locally commutative* at e if $e\Lambda e$ is commutative and that Λ is *locally commutative* if it is locally commutative at every primitive idempotent. Moreover, e is called *basic* if $e\Lambda$ is not isomorphic to any direct summand of $(1-e)\Lambda$. In this terminology, Λ is basic if and only if all its primitive idempotents are basic.

2.1. Theorem. *Let Λ be an artin algebra, and let e be a basic primitive idempotent in Λ such that Λ/J^2 is locally commutative at $e+J^2$. If S_e has finite projective or injective dimension, then* $\mathrm{Ext}^1_\Lambda(S_e, S_e) = 0$.

Proof. Firstly, we assume that S_e is of finite injective dimension. For proving that $\mathrm{Ext}^1_\Lambda(S_e, S_e) = 0$, it suffices to show that $eJe/eJ^2e = 0$. Let $a \in eJe$. Then $a + \Lambda(1-e)\Lambda \in [\Lambda_e, \Lambda_e]$ by Theorem 1.5. Since e is basic, $e\Lambda(1-e)\Lambda e \subseteq eJ^2e$. This yields an algebra homomorphism

$$f : \Lambda_e \to e\Lambda e/eJ^2e : x + \Lambda(1-e)\Lambda \mapsto exe + eJ^2e.$$

Thus, $a + eJ^2e = f(a + \Lambda(1-e)\Lambda)$ lies in the commutator group of $e\Lambda e/eJ^2e$. On the other hand, $e\Lambda e/eJ^2e \cong (e+J^2)(\Lambda/J^2)(e+J^2)$, which is commutative. Therefore, $a + eJ^2e = 0$, that is, $a \in eJ^2e$. The result follows in this case.

Next, assume that S_e has finite projective dimension. Let D be the standard duality between $\mathrm{mod}\Lambda$ and $\mathrm{mod}\Lambda^{\mathrm{op}}$. Then $D(S_e)$ is the simple Λ^{op}-module supported by the idempotent e^{o} corresponding to e, which is of finite injective dimension. Observe that the quotient of Λ^{op} modulo its radical square is also locally commutative at the class of e^{o} modulo the radical square. By what we have proven, $\mathrm{Ext}^1_\Lambda(S_e, S_e) \cong \mathrm{Ext}^1_{\Lambda^{\mathrm{op}}}(D(S_e), D(S_e)) = 0$. The proof of the theorem is completed.

Remark. The preceding result establishes the strong no loop conjecture for basic artin algebras Λ such that Λ/J^2 is locally commutative.

Now we shall specialize this result to finite dimensional algebras over a field. Recall that such an algebra is called *elementary* if its simple modules are all one dimensional over the base field; see [1].

2.2. Theorem. *Let Λ be a finite dimensional algebra over a field k, and let S be a simple Λ-module which is one dimensional over k. If S has finite projective or injective dimension, then* $\mathrm{Ext}^1_\Lambda(S, S) = 0$.

Proof. Let $e \in \Lambda$ be the primitive idempotent supporting S. Then Λ has a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents with $e = e_1$. We may assume that $e_1\Lambda, \ldots, e_r\Lambda$, with $1 \le r \le n$, are the non-isomorphic indecomposable projective modules in $\mathrm{mod}\Lambda$. Then

$$\Lambda/J \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $D_i = \mathrm{End}_\Lambda(e_i\Lambda/e_iJ)$ and n_i is the number of indices j with $1 \le j \le n$ such that $e_j\Lambda \cong e_i\Lambda$, for $i = 1, \ldots, r$. Now S is a simple $M_{n_1}(D_1)$-module, and hence $S \cong D_1^{n_1}$. Since S is one dimensional over k, it is one dimensional over D_1. In particular, $n_1 = 1$. That is, e is a basic primitive idempotent. Moreover, $e\Lambda e/eJe \cong Se \cong k$. Thus, for $x_1, x_2 \in e\Lambda e$, we can write $x_i = \lambda_i e + a_i$, where

$\lambda_i \in k$ and $a_i \in eJe$, $i = 1, 2$. This yields $x_1x_2 - x_2x_1 = a_1a_2 - a_2a_1 \in eJ^2e$. Therefore, $e\Lambda e/eJ^2e$ is commutative, and so is $(e + J^2)(\Lambda/J^2)(e + J^2)$. The result follows immediately from Theorem 2.1. The proof of the theorem is completed.

Remark. The preceding theorem establishes the strong no loop conjecture for finite dimensional elementary algebras, and hence for finite dimensional algebras over an algebraically closed field.

We shall extend our results in this direction. Let Λ be a finite dimensional elementary algebra over a field k. We may assume that $\Lambda = kQ/I$, where Q is a finite quiver, kQ is the path algebra, and I is an admissible ideal in kQ; see [1]. Recall that I is *admissible* if $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$ for some $n \geq 2$, where kQ^+ is the ideal in kQ generated by the arrows, and *monomial* if I in addition is generated by some paths. In this setting, the extension quiver of Λ is isomorphic to the quiver obtained from Q by shrinking the possible multiple arrows. If $p_1, \ldots, p_r$ are distinct paths in Q of length ≥ 2 from one vertex to another, then a k-linear combination

$$\rho = \lambda_1 p_1 + \cdots + \lambda_r p_r$$

is called a *minimal relation* for Λ if $\rho \in I$ and $\sum_{i \in J} \lambda_i p_i \notin I$ for any $J \subset \{1, \cdots, r\}$. Moreover, let $\sigma = \alpha_1\alpha_2 \cdots \alpha_r$ be an oriented cycle in Q, where the α_i are arrows. The support of σ, written as $\mathrm{supp}(\sigma)$, is the set of vertices in Q occurring as starting points of $\alpha_1, \ldots, \alpha_r$. The *idempotent supporting* σ is the sum of all primitive idempotents in Λ associated to the vertices in $\mathrm{supp}(\sigma)$. Write

$$\sigma_1 = \sigma,\ \sigma_i = \alpha_i \cdots \alpha_r \alpha_1 \cdots \alpha_{i-1},\ i = 2, \ldots, r,$$

called the *cyclic permutations* of σ. We shall say that σ is *cyclically free* in Λ if none of the σ_i with $1 \leq i \leq r$ is a summand of a minimal relation for Λ, and *cyclically non-zero* in Λ if none of the σ_i lies in I.

2.3. Theorem. *Let $\Lambda = kQ/I$ with Q a finite quiver and I an admissible ideal in kQ, and let σ be an oriented cycle in Q with supporting idempotent $e \in \Lambda$. If σ is cyclically free in Λ, then S_e has infinite projective and injective dimensions.*

Proof. Suppose that σ is cyclically free in Λ. If σ is a power of a shorter oriented cycle δ, then it is easy to see that δ is also cyclically free in Λ and $\mathrm{supp}(\delta) = \mathrm{supp}(\sigma)$. Hence, we may assume that σ is not a power of any shorter oriented cycle. Let $\sigma_1, \ldots, \sigma_r$, where $\sigma_1 = \sigma$, be the cyclic permutations of σ. It is then well known that the σ_i with $1 \leq i \leq r$ are pairwise distinct.

For any $p \in kQ$, denote by $\tilde{p}$ its class in Λ and by $\bar{p}$ the class of $\tilde{p}$ in Λ_e. Let W be the vector subspace of Λ_e spanned by the classes $\bar{p}$, where p ranges over the paths in Q different from $\sigma_1, \ldots, \sigma_r$. Then, there exist paths $p_1, \ldots, p_m$ in Q such that $\{\bar{p}_1, \ldots, \bar{p}_m\}$ is a k-basis of W. We claim that $\{\bar{\sigma}_1, \ldots, \bar{\sigma}_r, \bar{p}_1, \ldots, \bar{p}_m\}$ is a k-basis of Λ_e. Indeed, it clearly spans Λ_e. Assume that

$$\sum\nolimits_{i=1}^{r} \lambda_i \bar{\sigma}_i + \sum\nolimits_{j=1}^{m} \nu_j \bar{p}_j = \bar{0},\ \lambda_i, \nu_j \in k.$$

That is, $\sum \lambda_i \tilde{\sigma}_i + \sum \nu_j \tilde{p}_j \in \Lambda(1 - e)\Lambda$. Then

$$\sum\nolimits_{i=1}^{r} \lambda_i\, \tilde{\sigma}_i + \sum\nolimits_{j=1}^{m} \nu_j\, \tilde{p}_j = \sum\nolimits_{l=1}^{s} \mu_l\, \tilde{q}_l,\ \mu_l \in k,$$

where $q_1, \ldots, q_s$ are distinct paths in Q passing through a vertex not in $\mathrm{supp}(\sigma)$. Fix some t with $1 \leq t \leq r$. Letting ε_t be the trivial path in Q associated to the

starting point a_t of σ_t, we get

$$\sum_{i=1}^{r}\lambda_i\varepsilon_t\,\sigma_i\,\varepsilon_t+\sum_{j=1}^{m}\nu_j\,\varepsilon_t\,p_j\,\varepsilon_t-\sum_{l=1}^{s}\mu_l\varepsilon_t\,q_l\,\varepsilon_t\in I.$$

Note that the non-zero elements of the $\varepsilon_t\sigma_i\varepsilon_t$, $\varepsilon_t p_j\varepsilon_t$, $\varepsilon_t\,q_l\,\varepsilon_t\in kQ$ are distinct oriented cycles from a_t to a_t. Since σ is cyclically free in Λ, we have $\lambda_j=0$ whenever $\varepsilon_t\sigma_j\varepsilon_t$ is non-zero. In particular, $\lambda_t=0$. Therefore, the λ_i are all zero, and so are the ν_j. This proves our claim. Suppose now that $\bar\sigma\in[\Lambda_e,\Lambda_e]$. Then

$$\bar\sigma=\sum_{i=1}^{n}\eta_i(\bar u_i\bar v_i-\bar v_i\bar u_i) \tag{1}$$

where $\eta_i\in k$ and $u_i,v_i\in\{\sigma_1,\ldots,\sigma_r,p_1,\ldots,p_m\}$. For each $1\le i\le n$, we see easily that $u_iv_i\notin\{\sigma_1,\ldots,\sigma_r\}$ if and only if $v_iu_i\notin\{\sigma_1,\ldots,\sigma_r\}$, and in this case, $\bar u_i\bar v_i-\bar v_i\bar u_i\in W$. Therefore, the equation (1) becomes

$$\bar\sigma=\sum\eta_{ij}(\bar\sigma_i-\bar\sigma_j)+w, \tag{2}$$

where $\eta_{ij}\in k$ and $w\in W$. Let L be the linear form on Λ_e, which sends each of $\bar\sigma_1,\ldots,\bar\sigma_r$ to 1 and vanishes on W. Since $\sigma=\sigma_1$, applying L to the equation (2) yields $1=0$, a contradiction. Therefore, the class of $\bar\sigma$ in $\mathrm{HH}_0(\Lambda_e)$ is non-zero. Since $\bar\sigma$ lies in the radical of Λ_e, by Theorem 1.5, S_e has infinite projective and injective dimensions. The proof of the theorem is completed.

Example. Let $\Lambda=kQ/I$, where Q is the following quiver

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] \arrow[r, "\varepsilon"'] & 2 \arrow[l, bend left, "\beta"] \arrow[r, bend left, "\gamma"] & 3 \arrow[l, bend left, "\delta"] \arrow[r, bend left, "\mu"] & 4 \arrow[l, bend left, "\nu"]
\end{tikzcd}$$

and I is the ideal in kQ generated by $\alpha\beta,\delta\gamma,\beta\varepsilon,\varepsilon\beta,\nu\delta,\nu\mu,\mu\nu,\gamma\mu,\ \alpha\gamma\delta\beta\alpha\gamma-\varepsilon\gamma$. It is easy to see that the oriented cycle $\beta\alpha\gamma\delta$ is cyclically free in Λ. By Theorem 2.3, one of the simple modules S_1,S_2,S_3 has infinite projective dimension.

2.4. Corollary. *Let $\Lambda=kQ/I$ with Q a finite quiver and I an admissible ideal in kQ. If Q contains an oriented cycle which is cyclically free in Λ, then Λ has infinite global dimension.*

If I is a monomial ideal in kQ, then an oriented cycle in Q is cyclically free in Λ if and only if it is cyclically non-zero in Λ. This yields the following consequence, which can also be derived from results in [11].

2.5. Corollary. *Let $\Lambda=kQ/I$ with Q a finite quiver and I a monomial ideal in kQ. If Q contains an oriented cycle which is cyclically non-zero in Λ, then Λ has infinite global dimension.*

To conclude, we would like to draw the reader's attention to an even stronger version of the no loop conjecture as follows.

2.6. Extension Conjecture. *Let S be a simple module over an artin algebra. If $\mathrm{Ext}^1(S,S)$ is non-zero, then $\mathrm{Ext}^i(S,S)$ is non-zero for infinitely many integers i.*

This conjecture was originally posed under the name of *extreme no loop conjecture* in [14]. It remains open except for monomial algebras and special biserial algebras; see [6, 14].

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References

[1] M. Auslander, I. Reiten, and S. O. Smalø, "Representation Theory of Artin Algebras", Cambridge Studies in Advanced Mathematics 36 (Cambridge University Press, Cambridge, 1995).

[2] W. D. Burgess, K. R. Fuller, E. L. Green, and D. Zacharia, "Left monomial rings-a generalization of monomial algebras", Osaka J. Math. (3) 30 (1993) 543–558.

[3] W. D. Burgess and M. Saorín, "Homological aspects of semigroup gradings on rings and algebras", Canad. J. Math. (3) 51 (1999) 488–505.

[4] L. Diracca and S. Koenig, "Cohomological reduction by split pairs", J. Pure Appl. Algebra (3) 212 (2008) 471–485.

[5] E. L. Green, W. H. Gustafson, and D. Zacharia, "Artin rings of global dimension two", J. Algebra 92 (1985) 375 – 379.

[6] E. L. Green, Ø. Solberg, and D. Zacharia, "Minimal projective resolutions", Trans. Amer. Math. Soc. (7) 353 (2001) 2915 – 2939.

[7] D. Happel, "On the derived category of a finite dimensional algebra", Comment. Math. Helvetici. 62 (1987) 339 – 389.

[8] A. Happel, "Auslander-Reiten triangles in derived categories of finite-dimensional algebras", Proc. Amer. Math. Soc. 112 (1991) 641 – 648.

[9] A. Hattori, "Rank element of a projective module", Nagoya Math. J. 25 (1965) 113 – 120.

[10] K. Igusa, "Notes on the no loops conjecture", J. Pure Appl. Algebra (2) 69 (1990) 161 – 176.

[11] K. Igusa and D. Zacharia, "On the cyclic homology of monomial relation algebras", J. Algebra (2) 151 (1992) 502 – 521.

[12] B. T. Jensen, "Strong no-loop conjecture for algebras with two simples and radical cube zero", Colloq. Math. (1) 102 (2005) 1 – 7.

[13] H. Lenzing, "Nilpotente Elemente in Ringen von endlicher globaler Dimension", Math. Z. 108 (1969) 313 – 324.

[14] S. Liu and J. P. Morin, "The strong no loop conjecture for special biserial algebras", Proc. Amer. Math. Soc. (12) 132 (2004) 3513 – 3523.

[15] N. Marmaridis and A. Papistas, "Extensions of abelian categories and the strong no-loops conjecture", J. Algebra (1) 178 (1995) 1 – 20.

[16] C. Paquette, "Strictly stratified algebras revisited", Comm. Algebra (8) 37 (2009) 2591 – 2600.

[17] D. Skorodumov, "The strong no loop conjecture is true for mild algebras", preprint (arXiv:1011.1143v2).

[18] J. Stallings, "Centerless groups-an algebraic formulation of Gottlieb's theorem", Topology 4 (1965) 129 – 134.

[19] C. Weibel, "An introduction to homological algebra", Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, Cambridge, 1994).

[20] D. Zacharia, "Special monomial algebras of finite global dimension", NATO Adv. Sci. Inst. 233 (Kluwer, Dordrecht, 1988) 375 – 378.

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